

Monotonicity of p -norms of multiple operators via unitary swivels

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Abstract

Following the various statements of [DW16] to their logical conclusion, this note explicitly argues the following statement, implicit in [DW16]: for positive semi-definite operators C_1, \dots, C_L , a unitary V_{C_i} commuting with C_i , and $p \geq 1$, the quantity

$$\max_{V_{C_1}, \dots, V_{C_L}} \left\| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \right\|_p^p$$

is monotone non-increasing with respect to p . The idea from [DW16] is that by allowing unitary swivels connecting a long chain of positive semi-definite operators together, we can establish such a statement, which might not hold generally without the presence of the unitary swivels. Other related statements follow directly from [DW16] as well, being implicit there, and are given explicitly in this note.

1 Introduction

In this short note, I conduct the exercise of combining the various statements given in [DW16] and taking them to their logical conclusion. The result is a monotonicity inequality regarding p -norms of multiple operators strung together in a sequence. The only modification I make to the prior statements from [DW16] is to substitute density operators with general positive semidefinite operators. In [DW16], my coauthor and I were motivated by concerns in quantum information theory, and so there we worked exclusively with density operators (positive semi-definite operators with trace equal to one); however, it is obvious that all of the inequalities established there extend to the more general case when the operators are positive semi-definite with no restriction on their trace.

One of the main messages of [DW16] is that it is possible to establish non-trivial orderings of generalized Rényi entropies formed by connecting the marginals of density operators together in a product under a Schatten p -norm, while at the same time allowing for “unitary swivels” between these operators. In [DW16], my coauthor and I used the phrase “unitary swivels” to describe the method for arriving at the aforementioned inequalities, because, in spite of the fact that straightforward multi-operator extensions of the statements do not appear to be generally true, we showed how they hold if allowing for unitary swivels interleaved in a large chain of operators connected together. The bedrock upon which these results rested is the powerful method of complex

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interpolation [BL76], which has found a number of applications in a variety of areas in mathematics and physics.

To begin with, let us recall the following explicit statement from [DW16, Proposition 18], as specialized in [DW16, Corollary 19]:¹

$$\max_{V_{\rho_C}} \left\| \rho_{AC}^{1/p} V_{\rho_C} \rho_C^{-1/p} \rho_{BC}^{1/p} \right\|_p^p \text{ is monotone non-increasing for } p \geq 2, \quad (1)$$

where ρ_{ABC} is a density operator acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\rho_{BC} = \text{Tr}_A\{\rho_{ABC}\}$, $\rho_{AC} = \text{Tr}_B\{\rho_{ABC}\}$, and $\rho_C = \text{Tr}_{AB}\{\rho_{ABC}\}$ are its marginals, V_{ρ_C} is a unitary commuting with ρ_C , and $\|X\|_p \equiv [\text{Tr}\{|X|^p\}]^{1/p}$ is the Schatten p -norm of an operator X . In [DW16, Section 6], it was discussed how one can chain together various density operators acting on tensor-product Hilbert spaces and obtain results similar to those given in the rest of the paper [DW16]. Carrying this through, the conclusion is that the following statement holds

$$\max_{V_{C_1}, \dots, V_{C_L}} \left\| C_1^{1/p} V_{C_1} C_2^{1/p} V_{C_2} \dots C_L^{1/p} V_{C_L} \right\|_p^p \text{ is monotone non-increasing for } p \geq 2, \quad (2)$$

for C_1, \dots, C_L density operators and V_{C_i} a unitary commuting with C_i . In [DW16, Remark 12], it was mentioned how the optimizations over commuting unitaries can be replaced with more explicit bounds found by applying the Stein–Hirschman operator interpolation theorem [Ste56, Hir52]. Carrying this statement through as well, the conclusion is that the following inequality holds for $2 \leq q \leq p$:

$$\log \left\| C_1^{1/p} C_2^{1/p} \dots C_L^{1/p} \right\|_p^p \leq \int_{-\infty}^{\infty} dt \beta_{q/p}(t) \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \dots C_L^{(1+it)/q} \right\|_q^q, \quad (3)$$

if C_1, \dots, C_L are density operators and $\beta_{\theta}(t) \equiv \sin(\pi\theta)/(2\theta [\cosh(\pi t) + \cos(\pi\theta)])$, a probability distribution over $t \in \mathbb{R}$ and with a parameter $\theta \in [0, 1]$. In [DW16, Section 6], it was also discussed how one can obtain limits of the inequalities presented in the paper by applying the well known Lie-Trotter product formula. Carrying this through (i.e., taking the limit $p \rightarrow \infty$), the conclusion is that the following inequality holds

$$\log \text{Tr} \{ \exp \{ \log C_1 + \dots + \log C_L \} \} \leq \int_{-\infty}^{\infty} dt \beta_0(t) \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \dots C_L^{(1+it)/q} \right\|_q^q, \quad (4)$$

where $\beta_0(t) \equiv \lim_{\theta \rightarrow 0} \beta_{\theta}(t) = \pi/(2[\cosh(\pi t) + 1])$. By inspection of the proof given in [DW16, Proposition 18], it is clear that the inequalities in (3)–(4) hold for positive semi-definite operators as well. We can also see from that proof that (3) holds more generally for $1 \leq q \leq p$ and (4) for $1 \leq q$.

2 Explicit Proofs of (2)–(4)

In the rest of this note, I give explicit proofs of (2)–(4) for the benefit of the reader, following the steps outlined in [DW16] line by line.

¹Here and throughout, I am following the labeling in the arXiv post for [DW16].

Theorem 1 *Let C_1, \dots, C_L be positive semi-definite operators, let V_{C_i} denote a unitary commuting with C_i for all $i \in \{1, \dots, L\}$, and let $p \geq 1$. Then the following quantity is monotone non-increasing with respect to p :*

$$\max_{V_{C_1}, \dots, V_{C_L}} \left\| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \right\|_p^p. \quad (5)$$

Proof. The proof of this statement is essentially identical to the proof of [DW16, Proposition 18]. It is a consequence of a well known complex interpolation theorem recalled as Lemma 4 below. Let V_{C_1}, \dots, V_{C_L} denote a set of fixed unitaries, where V_{C_i} commutes with C_i . Let q be such that $1 \leq q < p$ (there is nothing to prove if $q = p$). For $z \in \mathbb{C}$, pick

$$G(z) = C_1^{z/q} V_{C_1} \cdots C_L^{z/q} V_{C_L}, \quad (6)$$

$$p_0 = \infty, \quad (7)$$

$$p_1 = q, \quad (8)$$

$$\theta = q/p, \quad (9)$$

the choices above being identical to those in [DW16, Eq. (7.6)-(7.9)]. This implies that $p_\theta = p$. Applying Lemma 4 gives

$$\|G(\theta)\|_p \leq \sup_{t \in \mathbb{R}} \|G(it)\|_\infty^{1-\theta} \sup_{t \in \mathbb{R}} \|G(1+it)\|_q^\theta. \quad (10)$$

Consider that

$$\|G(\theta)\|_p = \left\| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \right\|_p, \quad (11)$$

$$\|G(it)\|_\infty = \left\| C_1^{it/q} V_{C_1} \cdots C_L^{it/q} V_{C_L} \right\|_\infty \leq 1, \quad (12)$$

$$\|G(1+it)\|_q = \left\| C_1^{(1+it)/q} V_{C_1} \cdots C_L^{(1+it)/q} V_{C_L} \right\|_q \quad (13)$$

$$= \left\| C_1^{1/q} C_1^{it/q} V_{C_1} \cdots C_L^{1/q} C_L^{it/q} V_{C_L} \right\|_q \quad (14)$$

$$\leq \max_{W_{C_1}, \dots, W_{C_L}} \left\| C_1^{1/q} W_{C_1} \cdots C_L^{1/q} W_{C_L} \right\|_q, \quad (15)$$

which are conclusions identical to those in [DW16, Eq. (7.11)-(7.17)]. Putting everything together, we find that, for all V_{C_1}, \dots, V_{C_L} , the following inequality holds

$$\left\| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \right\|_p \leq \max_{W_{C_1}, \dots, W_{C_L}} \left\| C_1^{1/q} W_{C_1} \cdots C_L^{1/q} W_{C_L} \right\|_q^\theta, \quad (16)$$

which is equivalent to

$$\left\| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \right\|_p^p \leq \max_{W_{C_1}, \dots, W_{C_L}} \left\| C_1^{1/q} W_{C_1} \cdots C_L^{1/q} W_{C_L} \right\|_q^q. \quad (17)$$

Since (17) holds for all V_{C_1}, \dots, V_{C_L} , the statement of the theorem follows. ■

Theorem 2 *Let C_1, \dots, C_L be positive semi-definite operators, and let $p > q \geq 1$. Then the following inequality holds:*

$$\log \left\| C_1^{1/p} C_2^{1/p} \cdots C_L^{1/p} \right\|_p^p \leq \int_{-\infty}^{\infty} dt \beta_{q/p}(t) \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \cdots C_L^{(1+it)/q} \right\|_q^q. \quad (18)$$

Proof. Here we directly follow the suggestion from [DW16, Remark 12]. Pick $G(z)$, p_0 , p_1 , and θ as in (6)–(9), with $V_{C_1} = \dots = V_{C_L} = I$. Applying Lemma 5 below, we find that

$$\log \|G(\theta)\|_{p_\theta} \leq \int_{-\infty}^{\infty} dt \, \alpha_\theta(t) \log \|G(it)\|_{p_0}^{1-\theta} + \beta_\theta(t) \log \|G(1+it)\|_{p_1}^\theta. \quad (19)$$

After using that

$$\log \|G(it)\|_{p_0}^{1-\theta} = \log \left\| C_1^{it/q} \dots C_L^{it/q} \right\|_\infty^{1-\theta} \leq 0, \quad (20)$$

as recalled above, we are left with

$$\log \|G(\theta)\|_{p_\theta} \leq \int_{-\infty}^{\infty} dt \, \beta_\theta(t) \log \|G(1+it)\|_{p_1}^\theta. \quad (21)$$

This is then equivalent to the statement of the theorem. ■

Corollary 3 *Let C_1, \dots, C_L be positive definite operators, and let $q \geq 1$. Then the following inequality holds:*

$$\log \text{Tr} \{ \exp \{ \log C_1 + \dots + \log C_L \} \} \leq \int_{-\infty}^{\infty} dt \, \beta_0(t) \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \dots C_L^{(1+it)/q} \right\|_q^q. \quad (22)$$

Proof. Consider that

$$\left\| C_1^{1/2p} C_2^{1/2p} \dots C_L^{1/2p} \right\|_{2p}^{2p} = \text{Tr} \left\{ \left[C_L^{1/2p} \dots C_2^{1/2p} C_1^{1/p} C_2^{1/2p} \dots C_L^{1/2p} \right]^p \right\}. \quad (23)$$

Then by the multioperator Lie–Trotter product formula [Suz85], we have that

$$\lim_{p \rightarrow \infty} \text{Tr} \left\{ \left[C_L^{1/2p} \dots C_2^{1/2p} C_1^{1/p} C_2^{1/2p} \dots C_L^{1/2p} \right]^p \right\} = \text{Tr} \{ \exp \{ \log C_1 + \dots + \log C_L \} \}. \quad (24)$$

The inequality in the statement of the corollary is then a direct consequence of Theorem 2 and the above. ■

Lemma 4 *Let $S \equiv \{z \in \mathbb{C} : 0 \leq \text{Re} \{z\} \leq 1\}$, and let $L(\mathcal{H})$ be the space of bounded linear operators acting on a Hilbert space \mathcal{H} . Let $G : S \rightarrow L(\mathcal{H})$ be a bounded map that is holomorphic on the interior of S and continuous on the boundary.² Let $\theta \in (0, 1)$ and define p_θ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad (25)$$

where $p_0, p_1 \in [1, \infty]$. For $k = 0, 1$ define

$$M_k = \sup_{t \in \mathbb{R}} \|G(k+it)\|_{p_k}. \quad (26)$$

Then

$$\|G(\theta)\|_{p_\theta} \leq M_0^{1-\theta} M_1^\theta. \quad (27)$$

²A map $G : S \rightarrow L(\mathcal{H})$ is holomorphic (continuous, bounded) if the corresponding functions to matrix entries are holomorphic (continuous, bounded).

The following lemma is based on Hirschman's improvement of the Hadamard three-line theorem [Hir52].

Lemma 5 (Stein–Hirschman) *Let $S \equiv \{z \in \mathbb{C} : 0 \leq \operatorname{Re}\{z\} \leq 1\}$ and let $G : S \rightarrow L(\mathcal{H})$ be a bounded map that is holomorphic on the interior of S and continuous on the boundary. Let $\theta \in (0, 1)$ and define p_θ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad (28)$$

where $p_0, p_1 \in [1, \infty]$. Then the following bound holds

$$\log \|G(\theta)\|_{p_\theta} \leq \int_{-\infty}^{\infty} dt \, \alpha_\theta(t) \log \|G(it)\|_{p_0}^{1-\theta} + \beta_\theta(t) \log \|G(1+it)\|_{p_1}^\theta, \quad (29)$$

where $\alpha_\theta(t)$ and $\beta_\theta(t)$ are defined by

$$\begin{aligned} \alpha_\theta(t) &\equiv \frac{\sin(\pi\theta)}{2(1-\theta) [\cosh(\pi t) - \cos(\pi\theta)]}, \\ \beta_\theta(t) &\equiv \frac{\sin(\pi\theta)}{2\theta [\cosh(\pi t) + \cos(\pi\theta)]}. \end{aligned}$$

Remark 6 Fix $\theta \in (0, 1)$. Observe that $\alpha_\theta(t), \beta_\theta(t) \geq 0$ for all $t \in \mathbb{R}$ and we have

$$\int_{-\infty}^{\infty} dt \, \alpha_\theta(t) = \int_{-\infty}^{\infty} dt \, \beta_\theta(t) = 1, \quad (30)$$

(see, e.g., [Gra08, Exercise 1.3.8]) so that $\alpha_\theta(t)$ and $\beta_\theta(t)$ can be interpreted as probability density functions. Furthermore, the following limit holds

$$\lim_{\theta \searrow 0} \beta_\theta(t) = \frac{\pi}{2 [\cosh(\pi t) + 1]} = \beta_0(t), \quad (31)$$

where β_0 is also a probability density function on \mathbb{R} .

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